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Supergeometry and hermitian conjugation

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Abstract. *The theory of real smooth supermanifolds in the Berezin approach is not well adapted to the requirements of quantization since hermitian conjugation, which is the classical limit of taking operator adjoint in quantized theories, is not compatible with the sign rule of* $\mathbb{Z}_2$*-graded algebra. In this paper, we give the modifications of the theory necessary to reconcile it with hermitian conjugation. In particular, we introduce a second sign rule, and we show how to modify the usual definition of supermanifolds, getting hermitian supermanifolds. Almost all theorems of usual super differential geometry together with their proofs carry over to the new, hermitian setting. We also consider complex supermanifolds and hermitian metrics on complex vector bundles.*

1. INTRODUCTION

1.1. $\mathbb{Z}_2$-graded algebra versus quantum mechanics

The theory of real smooth supermanifolds in the Berezin approach (cf. in particular [2], [6], [7], [8], [9], [10], [11], [12]) has been developed by the guiding principle of maximal analogy with ordinary differential geometry. However, it turns out that this principle has led to a theory which while being mathematically aesthetic is not well adapted to the requirements of quantization. In fact, I want to show that we need a somewhat modified notion of supermanifolds both for modelling classical configuration spaces of

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fermionic degrees of freedom (cf. [13]) and for super space-time of supersymmetric theories formulated in superfields.

The source of the trouble is an apparent contradiction between the rules of $\mathbb{Z}_2$–graded algebra and that of quantum mechanics: if H is the Hilbert (or, may be, Krein) state space of a quantized system then it is the direct sum of the subspace H_0 of bosonic states with the subspace H_1 of fermionic ones. In fact, if the system has $SL(2, \mathbb{C})$ as symmetry group (as quantum field theory does) then H_0 , H_1 are the eigenspaces of $360°$ rotation (the axis of rotation does not matter). Now the familiar rule

$$(1.1.1) \qquad \langle u, v\rangle = \overline{\langle v, u\rangle}$$

is a flagrant violation of the sign rule of $\mathbb{Z}_2$-graded algebra - at least if this rule is applied over the ground field $\mathbb{R}$: in this case, a factor $(-1)^{|u||v|}$ on one side would be required.

This violation has consequences: if $A, B : H \to H$ are bounded operators then

$$(1.1.2) \qquad A^* B^* = (BA)^*$$

which again violates the sign rule!

This looks like a faux pas in the mathematical description of nature. However, the description of free field theory in Fock space is nowadays too well established to be reasonably questioned. So we have to reconcile our theory with the rules (1.1.1), (1.1.2). Indeed, this can be done in a rather satisfactory way, by modifying and augmenting the sign rule in such a way that (1.1.1), (1.1.2) are in perfect accordance with it. It turns out that the new rules (see 2.1 below) have the same consistency as the old rule.

1.2. Hermitian conjugation in supergeometry

Suppose that X is a smooth supermanifold (in the usual sense) the complex-valued superfunctions $f \in \mathcal{O}_X \otimes \mathbb{C}$ of which describe the classical observables of a system (e.g. a la [3]). The classical picture should arise by passing to the limit $h \to 0$; and, in the limit, the operation $A \to A^*$ on the operators of the quantum system should give rise to a skew-linear involutive map

$$(1.2.1) \qquad - : \mathcal{O}_X \otimes \mathbb{C} \to \mathcal{O}_X \otimes \mathbb{C}$$

which because of (1.1.2) has to satisfy the rule

$$(1.2.2) \qquad \overline{uv} = \bar{v}\bar{u}$$

We call (1.2.1) the *hermitian conjugation*. Under quantization, the $^-$ -invariant elements of $\mathcal{O}_X \otimes \mathbb{C}$ should correspond to self-adjoint (or, at least, symmetric) operators, i. e.

to genuine observables. Note that these elements do not form a subalgebra: Indeed, the product of two odd «real» (=invariant) elements is «imaginary», i. e. it changes sign under $^-$. For a mathematician, this may be psychologically hard to accept, but physics requires it: indeed, if two self-adjoint bounded operators in Hilbert space anticommute then their product is antiselfadjoint. Like anticommutativity of classical fermion fields (cf. [13]), the rule (1.2.2) has always been known and used by physicists, and it has been widely ignored by mathematicians.

For the reader who is willing to accept the thesis of the present author that the configuration space of a classical field theory with fermions should be viewed as infinite-dimensional supermanifold (cf. [13]), here is an impressive example supporting the above assertion: let us copy the standard energy-momentum tensor for a spin $1/2$ Dirac field ϕ from (say) the text-book [4], §5.2:

$$T^{\mu\nu} = \frac{i}{2}(\bar{\phi}\gamma^\mu\partial^\nu\phi - \partial^\nu\bar{\phi}\gamma^\mu\phi)$$

Both $T^{\mu\nu}$ and the standard free field Lagrangian $L = T^\mu_\mu - m\bar{\phi}\phi$ are real with respect to hermitian conjugation while if conjugating them with keeping the order of terms both expressions would be imaginary.

1.3. Why hermitian supergeometry?

The «obvious» conjugation on $\mathcal{O}_X \otimes \mathbb{C}$ (i.e. that which leaves $\mathcal{O}_X$ invariant) does not satisfy (1.2.2). One could try to define a hermitian supermanifold as a supermanifold equipped with a map (1.2.1) which satisfies (1.2.2). However, the distinction of $\mathcal{O}_X$ as subalgebra of $\mathcal{O}_X \otimes \mathbb{C}$ does not make any physical sense, and therefore it should not occur in the mathematical theory.

Now it is not hard to prove that for any skew-linear involutive even map (1.2.1) which satisfies (1.2.2), there exists at least locally an isomorphism

$$\mathcal{O}_X \otimes \mathbb{C} \to C^\infty_{\mathbb{C}}[\xi_1, \dots, \xi_n]$$

($C^\infty_{\mathbb{C}}$ is the sheaf of complex-valued smooth functions on the underlying manifold $\tilde{X}$) such that $^-$ acts by

$$\sum f_\mu(x)\xi^\mu \mapsto \sum(-1)^{|\mu|(|\mu|-1)/2}\overline{f_\mu(x)}\xi^\mu \tag{1.3.1}$$

These observations suggest to modify the usual definition of supermanifolds, with incorporating (1.3.1) into the local model. Actually, this modification is even necessary in view of the morphisms to be allowed. Let us look at a toy model (cf. also 3.6 for the physically interesting supersymmetry group where the situation is completely analogous):

If the linear supermanifold $X = L(1|2)$ has coordinates $(x|\xi, \eta)$ which we assume to be «real» (i.e. invariant under $^{-}$) then e. g.

$$\phi^*(x) = x + \xi\eta \qquad \phi^*(\xi) = \xi, \qquad \phi^*(\eta) = \eta$$

defines in the usual theory of supermanifolds (which we henceforth call *traditional*, to distinguish it from the hermitian theory to be developed in this paper) an automorphism $\phi : X \to X$. However, for a physicist, this statement is unacceptable since $\overline{\phi^*(x)} = x + \eta\xi = x - \xi\eta$; thus $\phi^*(x)$ is not «real» although x is! On the other hand, the setting $\phi^*(x) = x + i\xi\eta$ is physically acceptable but not implementable in the traditional theory of smooth smf's. This forces to modify the latter. Fortunately enough, these modifications can be made in a rather systematic way, and it is obvious in many cases what to do; also, almost all theorems of the traditional theory yield corresponding theorems in the hermitian setting, with the proofs carrying over without difficulty. Therefore, after developing hermitian $\mathbf{Z}_2$-graded algebra in detail, we can restrict ourselves onto a short account of the specific features of finite-dimensional hermitian supergeometry. In [15], [16], the infinite-dimensional case is treated.

It is interesting to note that among the people who worked on supercalculus questions, DeWitt was one of the very few who did not walk into the trap of «mathematical simplicity» concerning real structures: although in his book [5] he does not formulate the second sign rule (cf. 2.1 below), he actually works in the hermitian framework instead of the traditional one from the beginning. In particular, his «algebra of supernumbers» Λ_∞ is a hermitian algebra (to be defined below), and the «supervector space» introduced in [5], 1.4 is the same as a free hermitian module over Λ_∞.

2. HERMITIAN $\mathbb{Z}$-GRADED ALGEBRA

2.1. Hermitian vector spaces, sign rules

While traditional $\mathbf{Z}_2$-graded algebra (cf. the sources quoted above) makes sense over any ground field or ring (even the requirement that 2 be a unit could be dropped), hermitian $\mathbb{Z}_2$-graded algebra is bound to the specific situation over the complex number field $\mathbb{C}$.

We first recall the

Parity rule: the parity of an $\mathbf{R}$-multilinear expression is the sum of the parities of the terms involved in it.

Also, we adopt the sign rule in the following form:

First Sign Rule: whenever in a $\mathbb{C}$-multilinear expression standing on the r. h. s. of an equation two adjacent terms A, B are interchanged (with respect to their position on the r. h. s.) the sign $(-1)^{|A||B|}$ occurs. We stress the importance of the innocently looking word « $\mathbb{C}$-multilinear ». The terms in (1.1.1), (1.1.2), (1.2.2) as well as the hermitian conjugation to be introduced below are skew-linear!

A *hermitian vector space* is a $\mathbb{Z}_2$-graded complex vector space A equipped with a *hermitian structure*, i. e. a skew-linear involutive even map $^- : A \to A$. Then, setting

$$A_{\mathbf{R}} := \{a \in A : a = \bar{a}\}$$

(subspace of *real elements*) we have

$$A = A_{\mathbf{R}} \oplus \mathrm{i}\, A_{\mathbf{R}}$$

decomposition into real and imaginary part); this is a sum of $\mathbb{Z}_2$-graded vector spaces over $\mathbf{R}$. We call the elements of $A_{\mathbf{R}} \cup \mathrm{i}\, A_{\mathbf{R}}$ the *elements of definite reality.*

$\mathbb{C}$ itself will always be considered as hermitian vector space by $\mathbb{C}_1 := 0$, $^- :=$ usual complex conjugation.

The new ingredient of the theory is the second sign rule below. For its formulation we need some preparations: if $a_1, \ldots, a_n$ are homogeneous elements of (may be different) $\mathbb{Z}_2$-graded vector spaces we will write

$$\epsilon(a_1, \ldots, a_n) := (-1)^{\sum_{1 \le j < k \le n} |a_j||a_k|} . \tag{2.1.1}$$

If A is a $\mathbb{Z}_2$-commutative algebra then this is the sign generated by backward ordering:

$$a_n \ldots a_1 = \epsilon(a_1, \ldots, a_n) a_1 \ldots a_n .$$

(2.1.1) can be rewritten: let $N := \sum_{j=1}^{n} |a_j|$ be the number of indices j for which a_j is odd, and set

$$\epsilon_N := (-1)^{N(N-1)/2} .$$

Now $N(N-1) = 2\sum_{1 \le j < k \le n} |a_j||a_k| + \sum_{j=1}^{n}(|a_j|^2 - |a_j|)$; since $|a_j|$ takes the values 0 and 1 only, the second sum vanishes, and thus

$$\epsilon(a_1, \ldots, a_n) = \epsilon_N .$$

Obviously, ϵ_N depends only on the residue of N modulo 4, and $\epsilon_0 = \epsilon_1 =$, $\epsilon_2 = \epsilon_3 = -1$.

Now we may formulate the

Second Sign Rule: if conjugation is applied to a multilinear expression in the terms $a_1, \ldots, a_n$ (i.e. if conjugation is resolved into termwise conjugation), the expression acquires the sign factor $\epsilon(a_1, \ldots, a_n)$, in addition to the sign factors generated by re-ordering of terms.

Sometimes it makes sense (as in (1.2.2)), or it is even required (as in (1.1.1), (1.1.2)) to reorder the terms backwards. Then the sign $\epsilon(a_1, \ldots, a_N)$ cancels with the sign generated by the reordering, and thus (1.1.1), (1.1.2), (1.2.2) are in perfect accordance with both sign rules.

2.2. Hermitian algebras

A *non-associative hermitian algebra* is a hermitian vector space R together with an even bilinear composition law $(u,v) \mapsto uv$ such that the rule (1.2.2) holds. If R is moreover associative and has a unit we simply call it a *hermitian algebra*.

In Manin's terminology (cf. [9], 3.6.1), this is an algebra with a real structure of type $(1,1,1)$. Here we argue that in view of the requirements of physics, it plays a preferred role in comparison to the $2^3 - 1$ other types of real structures proposed in [9].

Note that the law (1.2.2) complies with the sign rules but is not canonically required by them! Alternatively, one could also require

$$\overline{uv} = \epsilon(u,v)\bar{u}\bar{v} \tag{2.2.1}$$

Of course, if R is $\mathbb{Z}_2$-commutative then (2.2.1) and (1.1.2) are equivalent.

If R is a hermitian algebra and $a_1, \ldots, a_n \in R$ then one has by induction

$$\overline{a_1 \ldots a_n} = \overline{a_n}\,\overline{a_{n-1}} \ldots \overline{a_1}\,. \tag{2.2.2}$$

(This is also true in the non-associative case where the l. h. s. contains brackets: then the r. h. s. is to be equipped with brackets arranged mirror-like).

If R is moreover $\mathbb{Z}_2$-commutative then (2.2.2) rewrites to

$$\overline{a_1 \ldots a_n} = \epsilon(a_1, \ldots, a_n)\overline{a_1} \ldots \overline{a_n}\,.$$

Note that if u, v are real elements of definite parity in a hermitian $\mathbb{Z}_2$-commutative algebra then $\overline{uv} = \epsilon(u,v)uv$. Thus uv is real if u or v is even; if both are odd it is imaginary (!). Hence, *the real elements of a hermitian algebra do not form a subalgebra.* Thus, while a hermitian vector space is in essence just a «blown-up» (i.e. complexified) real $\mathbb{Z}_2$-graded vector space, a hermitian algebra is *not* a complexified real algebra (unless its odd part vanishes), and therefore we prefer the term «hermitian structure» instead of «real structure».

A *homomorphism of hermitian algebras* $\phi : R \to S$ is a homomorphism of algebras with unit which as linear map is both even and real.

2.3. Hermitian modules

A *hermitian module* A over a hermitian algebra R is a $\mathbb{Z}_2$-graded bimodule over that algebra together with a hermitian structure on its underlying vector space such that

$$\overline{ua} = \bar{a}\bar{u} \tag{2.3.1}$$

for $u \in R$, $a \in A$. Applying here hermitian conjugation and substituting a for $\bar{a}$ and u for $\bar{u}$, respectively, we get

$$\bar{u}\bar{a} = \overline{au};$$

thus, we have perfect left-right symmetry.

If R is $\mathbb{Z}_2$-graded commutative, all left modules are bimodules in the usual way, and then (2.3.1) rewrites to $\overline{ua} = \epsilon(a,u)\bar{u}\bar{a}$.

We also note that given a hermitian algebra R and a hermitian vector space A which is equipped with the structure of a right module over the $\mathbb{Z}_2$-graded algebra R then, setting $ua := \overline{\bar{a}\bar{u}}$, A becomes a hermitian module. However, if R is $\mathbb{Z}_2$-commutative it may happen that left and right module structure are not connected in the usual way.

The hermitian modules over a fixed hermitian algebra form an additive category, the morphisms being just the ordinary bimodule homomorphisms.

Let A, B be hermitian modules. Then the vector space $\mathrm{Hom}_R(A,B)$ is an R-bimodule by $(u\phi)(a) := u\phi(a), (\phi u)(a) := \phi(ua)$. The second sign rule suggests to make $\mathrm{Hom}_R(A,B)$ a hermitian vector space:

$$\bar{\phi}(a) = \epsilon(\phi,a)\overline{\phi(\bar{a})} \quad \text{for} \quad \phi \in \mathrm{Hom}(A,B), \quad a \in A\,.$$

The definition is correct: if $u \in R$ then

$$\begin{aligned}\bar{\phi}(ua) &= \epsilon(\phi,u)\epsilon(\phi,a)\overline{\phi(\bar{a}\bar{u})} = \epsilon(\phi,u)\epsilon(\phi,a)\overline{\phi(\bar{a})\bar{u}}\\ &= \epsilon(\phi,u)\epsilon(\phi,a)u\overline{\phi(\bar{a})} = \epsilon(\phi,u)u\bar{\phi}(a);\end{aligned}$$

i. e. $\bar{\phi}$ is a homomorphism of left modules. Similarly,

$$\begin{aligned}\bar{\phi}(au) &= \epsilon(a,\phi)\epsilon(u,\phi)\overline{\phi(\bar{u}\bar{a})} = \epsilon(a,\phi)\overline{\bar{u}\phi(\bar{a})}\\ &= \epsilon(a,\phi)\overline{\phi(\bar{a})}u = \bar{\phi}(a)u\end{aligned}$$

i. e. $\bar{\phi}$ is also a homomorphism of right modules. Thus, $^-$ maps indeed $\mathrm{Hom}_R(A,B)$ into itself. With similar tedious but easy calculations one checks that $\mathrm{Hom}_R(A,B)$ is actually a hermitian module. From now on, we omit the proofs of consistency assertions like the above one.

Note that if $\phi : A \to B$ is a homomorphism then $\mathrm{Ker}(\phi)$, $\mathrm{Im}(\phi)$, $\mathrm{Coker}(\phi)$ will be hermitian modules only in the case that ϕ has both definite parity and definite reality. Note also that if A is generated over R by a subset S of real elements, and ϕ is even, then ϕ is real iff $\phi(S) \subseteq B_{\mathbf{R}}$.

For compositions of module homomorphisms $\phi : A \to B$, $\psi : B \to C$ we find $\overline{\psi\phi} = \epsilon(\psi,\phi)\bar{\psi}\bar{\phi}$, as to be expected.

Note that $\mathrm{Hom}_R(A,A)$, although being both a hermitian vector space and a $\mathbb{Z}_2$-graded algebra, is not a hermitian algebra: the composition of endomorphism satisfies (2.2.1) instead of (1.1.2). (However, if $S \subseteq \mathrm{Hom}_R(A,A)$ is a hermitian subspace which is at the same time a $\mathbb{Z}_2$-commutative subalgebra then S is a hermitian subalgebra).

We declare the *parity shift* $\Pi : A \to \Pi A$ to be a real, odd, bijective homomorphism. Thus $\overline{\Pi a} = (-1)^{|a|}\Pi\bar{a}$ for $a \in A$. A formal calculation yields that $\Pi\Pi : A \to \Pi\Pi A$ is imaginary. Indeed, one checks that the map $\Pi\Pi A \to A$, $\Pi\Pi a \mapsto i\,a$, is an isomorphism, and because of its functoriality it can be viewed as identification.

Let $A_1, \ldots, A_n$ be hermitian modules over $\mathbb{Z}_2$-commutative hermitian algebra R, and let the connection between left and right module structure be the usual one. Then we can turn $A_1 \otimes_R \ldots \otimes_R A_n$ into a hermitian module, too:

$$\overline{a_1 \otimes \ldots \otimes a_n} \mapsto \epsilon(a_1, \ldots, a_n)\overline{a_1} \otimes \ldots \otimes \overline{a_n}. \tag{2.3.2}$$

In order to see that this is correct, the usual universal property of the tensor product cannot be applied, due to the lack of R-multilinearity of the r. h. s. However, constructing the tensor product by factorizing the free R-module F generated by the set $A_1 \times \ldots \times A_n$ by the usual relations, it is easy to check that the map $F \to F$,

$$(a_1, \ldots, a_n) \mapsto \epsilon(a_1, \ldots, a_n)(\overline{a_1}, \ldots, \overline{a_n}),$$

leaves the submodule of relations invariant. Therefore, (2.3.2) is well-defined.

We call $A_1 \otimes_R \ldots \otimes_R A_n$ the *hermitian tensor product* of $A_1, \ldots, A_n$. The hermitian tensor product is associative and commutative in an obvious sense. We also note that the evaluation map $A \otimes \mathrm{Hom}(A, B) \to B$ is real.

Let R, S be hermitian algebras. Equipping $R \otimes S$ with the hermitian tensor product structure and with the usual algebra structure $(r \otimes s)(r' \otimes s') := \epsilon(r', s)rr' \otimes ss'$ for $r, r' \in R$, $s, s' \in S$, it becomes a hermitian algebra, too. Let $S \to R$ be a homomorphism of hermitian algebras, and A a hermitian S-module. Then $A \otimes_S R$ is a right R-module and a hermitian vector space; by the remarks in 2.3, it becomes a hermitian R-module. Setting here $S := \mathbb{C}$, $A := \mathbb{C}^{m|n}$, one gets a canonical hermitian structure on

$$\mathbb{C}^{m|n} \otimes R = R^{m|n}. \tag{2.3.3}$$

Let $e_1, \ldots, e_{m+n}$ be the real standard base in $\mathbb{C}^{m|n}$. Then conjugation acts by $\overline{\sum e_i u_i} = \sum \overline{u_i} e_i$. We call a hermitian R-module *free of rank* $m|n$ if it is isomorphic to (2.3.3).

The tensor algebra $T_R A = \bigoplus_{k \geq 0} \bigotimes^k A$ of a hermitian module over a hermitian algebra becomes a hermitian algebra again by the setting

$$\overline{a_1 \otimes \ldots \otimes a_n} = \overline{a_n} \otimes \ldots \otimes \overline{a_1} \tag{2.3.4}$$

for $a_1, \ldots, a_n \in A$. (By similar arguments as above, this is well-defined; here $\mathbb{Z}_2$-commutativity of R is not necessary.) Note that this is a completely different hermitian structure in comparison with that of the hermitian tensor product! The latter would yield a $\mathbb{Z}_2$-graded algebra which satisfies (2.2.1). Thus, in contradistinction to the question of whether a given composite expression is even or odd, the question whether it is real or imaginary is not always canonically answered by general, metamathematical rules.

We also call $\bigotimes^k A$ together with the hermitian structure (2.3.4) the k-th *hermitian tensor power* of A. Now the ideal $I \subseteq T_R A$ generated by all elements $ab - \epsilon(a,b)ba$, with a, b running through $A_0 \cup A_1$, is stable under conjugation, and hence the factor algebra $S_R A = T_R A/I$ (*symmetric algebra of A over R*) is a hermitian algebra again. It is not hard to show that $S_R A$ is $\mathbb{Z}_2$-commutative.

If S is any hermitian algebra over R, any real, even R-linear map $\phi : A \to S$ has a unique continuation to a homomorphism of hermitian algebras $T_R A \to S$. Moreover, if $[\phi(a), \phi(a')] = 0$ for all $a, a' \in A$ then the arising homomorphism $S_R A \to S$ is hermitian, too.

If R is a hermitian algebra and $(x|\xi) = (x_1, \ldots, x_m|\xi_1, \ldots, \xi_n)$ is a sequence of even and odd formal variables then the polynomial algebra $R[x|\xi]$ as well as the formal power series algebra $R[[x|\xi]]$ have unique structures of hermitian algebras such that all x_i and ξ_j are real. Explicitely,

$$\overline{\sum a_{\mu\nu} x^\mu \xi^\nu} = \sum \epsilon_{|\nu|+|a_{\mu\nu}|} \overline{a_{\mu\nu}} x^\mu \xi^\nu, \quad a_{\mu\nu} \in R.$$

2.4. Complex structures

Although a hermitian vector space is a complex vector space by definition, it conceptually plays the role of a real $\mathbb{Z}_2$-graded vector space. The role of complex $\mathbb{Z}_2$-graded vector spaces is taken over by hermitian vector spaces E equipped with a *complex structure*, i. e. an even, real map $J : E \to E$ with $J^2 = -1$ (note that viewing J as multiplication with i would lead to confusion!). Setting $E_\pm := Ker(J \mp i)$, we get a decomposition into mutually conjugated complex $\mathbb{Z}_2$-graded vector spaces:

$$E = E_+ \oplus E_-, \quad E_- = \overline{E_+}. \tag{2.4.1}$$

Conversely, given such a decomposition (2.4.1) of a hermitian vector space, it determines a complex structure on E in an obvious way.

Of course, if R is a hermitian algebra, a complex structure on a hermitian R-module A is a complex structure J on the underlying hermitian vector space of A which is R-linear. This is equivalent with saying that A_+, A_- are (non-hermitian) R-submodules.

In the following, let R be a $\mathbb{Z}_2$-commutative hermitian algebra. The k-th symmetric power $S^k A := S^k_R A$ of a free hermitian R-module with complex structure inherits a *type decomposition*: Setting

$$S^{i,j} A := S^i A_+ \otimes S^j A_-$$

for $i, j \geq 0$, we have

$$S^k A = \bigoplus_{i+j=k} S^{i,j} A .$$

Note that the $S^{i,j} A$ are non-hermitian R-modules. One has $\overline{S^{i,j} A} = S^{j,i} A$ for all i, j.

Let A, B be R-modules which are equipped with complex structures. We set

$$A \otimes_{R,J} B := A_+ \otimes B_+ \oplus A_- \otimes B_- \tag{2.4.2}$$

(*complex tensor product*),

$$\begin{aligned} \operatorname{Hom}_{R,J}(A, B) &:= \operatorname{Hom}(A_+, B_+) \oplus \operatorname{Hom}(A_-, B_-) \\ &= \{\phi \in \operatorname{Hom}_R(A, B) : \phi J_A = J_B \phi\} \end{aligned} \tag{2.4.3}$$

(*complex Hom*),

$$S^k_{R,J} A := S^{k,0} A_- \oplus S^{0,k} A_+ \tag{2.4.4}$$

(*complex symmetric power*). All three modules (2.4.2), (2.4.3), (2.4.4) carry obvious structures of hermitian R-modules with complex structures. For the motivation of these settings, cf. 4.2. Note also that the dual $A^* = A^*_+ \oplus A^*_-$ carries a complex structure, too.

Now let V be a complex $\mathbb{Z}_2$-graded vector space (without hermitian structure). As usual, we denote by $\bar{V}$ the *complex conjugated space*: it consists of all symbols $\bar{v}$ with v running through V; parity and module structure are fixed by requiring that the bijection $V \to \bar{V}, v \mapsto \bar{v}$, be skew-linear and even.

Now if V as above then $V \oplus \bar{V}$ is a hermitian vector space which is equipped with a complex structure; the action of $^-$ is suggested by the notations: $\overline{u + \bar{v}} \mapsto \bar{u} + v$, and $V = (V \oplus \bar{V})_+$, $\bar{V} = (V \oplus \bar{V})_-$.

If R is a hermitian algebra, and E_+ is an R-module (without hermitian structure) then, setting $E_- := \overline{E_+}$ and $E := E_+ \oplus E_-$, E becomes a hermitian R-module with complex structure. It follows that the assignment $E \to E_+$ establishes an equivalence between

(a) the category of hermitian R-modules with complex structure; the morphisms are the real R-homomorphisms which commute with the complex structure, and

(b) the category of R-modules (without hermitian structure).

2.5. Super Hilbert space

In essence, a super Hilbert space is an ordinary Hilbert space with a fixed decomposition into two closed subspaces. However, in order to adapt this to the hermitian calculus, we reformulate this:

A *super Hilbert space* is a complex $\mathbb{Z}_2$-graded vector space $H = H_0 \oplus H_1$ together with a $\mathbb{C}$-bilinear pairing

$$\bar{H} \times H \to \mathbb{C}, \quad (\bar{a}, b) \mapsto \langle \bar{a} | b \rangle$$

such that

(i)

$$\overline{\langle \bar{a} | b \rangle} = \langle \bar{b} | a \rangle \quad \text{for} \quad a, b \in H, \tag{2.5.1}$$

(ii) $\langle \bar{a} | a \rangle > 0$ for $a \in H, a \neq 0$,

(iii) H is complete with respect to the topology defined by the norm $a \mapsto ||a|| := \langle \bar{a} | a \rangle^{1/2}$.

REMARKS. (1) Obviously, (2.5.1) complies with the sign rule.

(2) We follow the convention of physical literature: The scalar product $(a, b) \mapsto \langle \bar{a} | b \rangle$ is $\mathbb{C}$-linear in the second argument. The notation introduced here is somewhat unusual but (as I hope) suggestive and helpful in avoiding confusion.

(3) The hermitian vector space with complex structure $H' := H \oplus \bar{H}$ (cf. 2.4) carries moreover two bilinear forms

$$\langle .,. \rangle_\pm : H' \times H' \to \mathbb{C},$$
$$\langle a + \bar{b}, c + \bar{d} \rangle_\pm := \langle \bar{b}, c \rangle \pm \epsilon(d, a) \langle \bar{d}, a \rangle .$$

While $\langle .,. \rangle_+$ is $\mathbb{Z}_2$-symmetric, $\langle .,. \rangle_-$ is $\mathbb{Z}_2$-alternating. Note the identity $\langle J(a + \bar{b}), J(c + \bar{d}) \rangle_\pm = -\langle a + \bar{b}, c + \bar{d} \rangle_\pm$. Also, if we view $\langle .,. \rangle_\pm$ as linear maps $H' \otimes H' \to \mathbb{C}$, the left-hand side being equipped with the tensor product hermitian structure, then $\langle .,. \rangle_+$ is real while $\langle .,. \rangle_-$ is imaginary.

(4) It is no incidence that the whole structure strongly resembles that of the complex tangent bundle of a Kähler manifold (cf. also 4.3 – 4.6). The apparent antiunitarity of J looks strange, but it is only due to our way of writing the pairing $\langle .,. \rangle_\pm$ (indeed, introducing temporarily the notation $\{a + \bar{b}, c + \bar{d}\}_\pm := \overline{\langle a + \bar{b}, c + \bar{d} \rangle_\pm}$, which is closer to classical treatment, we find unitarity: $\{J(a + \bar{b}), J(c + \bar{d})\}_\pm = \{a + \bar{b}, c + \bar{d}\}_\pm$).

Now let $D \subseteq H$ be a dense $\mathbb{Z}_2$-graded linear subspace of a super Hilbert space, and denote by $Op(H, D)$ the linear space of all linear maps $A : D \to D$ the adjoint of which maps D into itself, too. It is well known that any such A is closable, and that $Op(H) := Op(H,H)$ is just the set of bounded operators in H.

Obviously, $Op(H,D)$ is an algebra under composition. Moreover, taking the usual operator adjoint $A \mapsto A^*$,

$$\text{(2.5.2)} \qquad \langle \overline{A^* u} | v \rangle = \langle \bar{u} | Av \rangle \quad \text{for} \quad u, v \in D\,,$$

as hermitian conjugation, $Op(H,D)$ becomes a hermitian algebra. Obviously, (2.5.2) complies with the sign rules.

2.6. Brackets and Lie superalgebras

The *bracket (commutator)* in a hermitian superalgebra R is defined as usual:

$$\text{(2.6.1)} \qquad [u,v] := uv - \epsilon(u,v)vu\,.$$

One has the identity

$$\text{(2.6.2)} \qquad \overline{[u,v]} = [\bar{v},\bar{u}] = -\epsilon(u,v)[\bar{u},\bar{v}]\,.$$

Thus, we define in accordance with 2.2 a *hermitian Lie superalgebra* $\mathfrak{g}$ as a complex Lie superalgebra together with a hermitian structure such that the identity (2.6.2) holds. It follows that any associative hermitian algebra R if equipped with the bracket (2.6.1) becomes a hermitian Lie superalgebra. In particular, this applies to $Op(H,D)$ where H is a super Hilbert space.

Of course, a homomorphism (alias representation) of hermitian Lie super algebras is just a a homomorphism of Lie super algebras which as linear map is real (that is, it commutes with conjugation). Thus, if $\mathfrak{g} \to Op(H,D)$ is a representation of a hermitian Lie superalgebra in super Hilbert space H real elements will pass to symmetric operators.

Note that in any hermitian Lie superalgebra, *the bracket of real even elements is purely imaginary.* Under a representation $\mathfrak{g} \to Op(H,D)$, this corresponds to the fact that that the usual commutator of symmetric operators in Hilbert space is antisymmetric.

We also note that if $\mathfrak{g}$ is a real Lie algebra, and if $\mathfrak{g} \otimes \mathbb{C}$ is equipped with the complexified bracket and that hermitian conjugation which makes $\mathfrak{g}$ the subspace of imaginary (!) elements then $\mathfrak{g} \otimes \mathbb{C}$ will be a hermitian Lie superalgebra.

Let $\mathfrak{g}$ be a hermitian Lie superalgebra. One finds $\overline{ad_a} = ad_{\bar{a}}$ for $a \in \mathfrak{g}$. Thus, $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is an even, imaginary linear map. One can also define a «skew Killing form» $(a,b) \mapsto B(a,\bar{b}) := \mathrm{Str}(ad_a ad_{\bar{b}})$. Then $B(b,\bar{a}) = \overline{B(a,\bar{b})}$, and thus we get classically hermitian forms both on $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

Let R be a $\mathbf{Z}_2$-commutative hermitian algebra, and let A, B be free hermitian R-modules with bases $(e_i)_{i=1,\dots,m+n}, (f_j)_{j=1,\dots,r+s}$, respectively. To any homomorphism $\phi : A \to B$ we can assign the matrix (ϕ_{ji}) given by $\phi(e_i) = \sum f_j \phi_{ji}$. One calculates

$$\bar{\phi}_{ji} = \begin{cases} (-1)^{|f_j|(1-|e_i|)} \overline{\phi_{ji}} & \text{for } |\phi| = 0, \\ (-1)^{|e_i|(1-|f_j|)} \overline{\phi_{ji}} & \text{for } |\phi| = 1 . \end{cases}$$

Thus, ϕ is is real iff its matrix has the block structure $\begin{pmatrix} A & B \\ iC & D \end{pmatrix}$ for even ϕ, $\begin{pmatrix} A' & iB' \\ C' & D' \end{pmatrix}$ for odd ϕ, respectively; here $A, B, C, D, A', B', C', D'$ have real entries.

From this, it immediately follows that the *supertrace* $\mathrm{Str}(\phi)$ of any real R-module endomorphism ϕ is real, and that the even homomorphism $\mathrm{Str} : \mathrm{End}(A) \to R$ is real. Also, if ϕ is even, real, and invertible then its Berezinian $\mathrm{Ber}(\phi) \in R_0$ is real.

REMARK. (1) Almost certainly, there exist more consistency properties of the sign rules (which are, like the sign rules itself, of metamathematical nature) than I was able to check and list up here. Up to now, no case of serious inconsistency has been observed.

(2) One should also keep in mind that the sign rules are of metamathematical nature; they are useful in finding and memorizing the right definition and results, but not in «proving» anything. For instance, if we had choosen in 2.5 the mathematics' convention (i.e. the scalar product is $\mathbb{C}$-linear in the first argument) the sign rules would suggest another hermitian structure on $Op(H, D)$: $\bar{A}_{wrong} = (-1)^{|A|} \bar{A}$.

3. HERMITIAN SUPERMANIFOLDS

3.1. Hermitian superdomains

In the following, a *hermitian ringed space* $(X, \mathcal{O})$ consists of a Hausdorff paracompact topological space space (X) and a sheaf of hermitian algebras $\mathcal{O}$ on it.

A *hermitian superdomain* is a hermitian ringed space of the form

(3.1.1) $$(U, C^\infty_{\mathbb{C}}[\xi_1, \ldots, \xi_n])$$

with hermitian structure given by (1.3.1). The sequence $(x_1, \ldots, x_m | \, \xi_1, \ldots, \xi_n)$ of elements of $\mathcal{O}_{\mathbf{R}}$ will be referred to as *standard coordinate system*. Now

$$\mathcal{O}_{\mathbf{R},0} = \{ \sum_{|\mu| \equiv 0 \ (4)} F_\mu(x) \xi^\mu + \mathrm{i} \sum_{|\mu| \equiv 2 \ (4)} F_\mu(x) \xi^\mu \},$$

$$\mathcal{O}_{\mathbf{R},1} = \{ \sum_{|\mu| \equiv 1 \ (4)} F_\mu(x) \xi^\mu + \mathrm{i} \sum_{|\mu| \equiv 3 \ (4)} F_\mu(x) \xi^\mu \}$$

with $F_\mu(x)$ being real-valued smooth functions.

3.2. Hermitian supermanifolds

A *hermitian supermanifold* (*hermitian* smf) X is a hermitian ringed space which is locally isomorphic to a hermitian superdomain. The following assertions carry over together with their proofs from the traditional situation:

(i) One can view any smooth manifold Y, dim $Y =: m$, as hermitian supermanifold of dimension $m|0$, the structure sheaf being $C^\infty_{\mathbb{C}}$ with complex conjugation. It is easy to show that this makes the category of smooth manifolds a full subcategory of the category of hermitian supermanifolds.

On the other hand, if X is any hermitian supermanifold then the underlying space $\tilde{X}$ has a unique structure of a smooth manifold such that all local isomorphisms to hermitian superdomains have smooth underlying maps. Moreover, there is a unique morphism of hermitian algebra sheaves

$$\beta : \mathcal{O} \to C^\infty_{\mathbb{C}} .$$

β is surjective also on global sections, and its kernel is the sheaf $\mathcal{O}^1$ of nilpotents of $\mathcal{O}$. β commutes with morphisms, and it gives rise to an embedding of ringed spaces $\tilde{X} \to X$.

Note that one could substitute real-analytic for smooth functions almost throughout, getting analytic hermitian supermanifolds. On the other hand, the relation of hermitian smf's to complex-analytic smf's is somewhat more subtle and will be discussed in 4.

3.3. Morphisms

Let U be a hermitian superdomain with standard coordinates $(x|\xi)$, and let Y be any hermitian smf.

LEMMA. *Given a sequence $(x^*|\xi^*)$ of even and odd elements of $\mathcal{O}(Y)$ such that*

(3.3.1) $$\beta(x^*)(p) \in U \quad \text{for all} \quad p \in Y,$$

there is a unique morphism of ringed spaces $\phi : Y \to U$ such that $(x^*|\xi^*) = \phi^*(x|\xi)$. Moreover, ϕ is a morphism of hermitian ringed spaces iff all x_i^*, ξ_j^* are real.

Q.E.D. ∎

The proof copies that of the traditional theorem that supermanifold morphisms are determined by the coordinate pullbacks. Note that the condition (3.3.1) is not an «open» one: A priori, $\beta(x^*)(p) \in \mathbb{C}^m$.

Using the Lemma the following is not hard to prove:

COROLLARY. 3.3.1 *Let the structure sheaf of the ringed space (3.1) be equipped with some hermitian conjugation which turns it into a sheaf of hermitian algebras. Then, with this hermitian conjugation, (3.1) is actually a hermitian supermanifold.* Q.E.D. ∎

Thus, there is no loss of generality in taking the hermitian conjugation (1.3.1).

From the Lemma, we also get the description of morphisms by coordinate changes:

COROLLARY. 3.3.2 *Let U, Y be as above. Given a sequence $(x^*|\xi^*)$ of even and odd real elements of $\mathcal{O}(Y)$ such that (3.3.1) holds, there is a unique morphism $\phi : Y \to U$ such that $(x^*|\xi^*) = \phi^*(x|\xi)$.* Q.E.D. ∎

We call a sequence $(x^*|\xi^*)$ of even and odd elements of $\mathcal{O}_{\mathbf{R}}(U)$, $U \subseteq X$ open, a *real coordinate system* if U (as hermitian ringed space) is isomorphic to a hermitian superdomain such that under this isomorphism, $(x^*|\xi^*)$ passes into the standard coordinate system.

However, for studying supersymmetric field theories, the following notion is better adapted: we call a sequence of even and odd elements $(x|\theta)$ a *chiral coordinate system* if $(x|\,\mathrm{Re}\,(\theta),\ \mathrm{Im}\,(\theta))$ is a real coordinate system. Obviously, chiral coordinate systems exist locally iff the odd dimension is an even number.

COROLLARY. 3.3.3 *Let Y, U be as above, and let $(x|\theta)$ be a chiral coordinate system on U. Given a sequence $(x^*|\theta^*)$ of even and odd elements of $\mathcal{O}(Y)$ such that the x_i^* are real and (3.3.1) holds, there is a unique morphism $\phi : Y \to U$ such that $(x^*|\theta^*) = \phi^*(x|\theta)$.* Q.E.D. ∎

3.4. Classification

Let X be a hermitian supermanifold, $m|n := \dim X$, and set $\Phi := \mathcal{O}^1/\mathcal{O}^2$ where $\mathcal{O}^2$ is the square of $\mathcal{O}^1$ as ideal sheaf. Then Φ is a locally free hermitian $C^\infty_{\mathbb{C}}$-module sheaf of rank $0|n$. Hence, the subsheaf $\Phi_{\mathbf{R}}$ of real sections is a locally free C^∞-module sheaf of rank $0|n$, that is, it is the section sheaf of a real vector bundle. The hermitian version of Batchelor's Classification Theorem is: there exists an isomorphism of hermitian algebra sheaves $\mathcal{O} \cong S\Phi$ (symmetric power over $C^\infty_{\mathbb{C}}$). Hence, hermitian supermanifolds are classified by the same data as traditional ones, namely by real vector bundles. Nevertheless, a natural functor between both categories does not seem to exist.

3.5. Linear hermitian supermanifolds, supergroups, vector bundles

Let V be a real $\mathbb{Z}_2$-graded vector space, and let $V_{\mathbb{C}}$ be its complexification viewed as hermitian vector space. The *linear hermitian supermanifold* $L(V)$ associated to V has underlying topological space V_0 and structure sheaf $C^\infty_{\mathbb{C}} \otimes S_{\mathbb{C}} V^*_{\mathbb{C},1}$. $L(V)$ is the representing object for the cofunctor

$$\{\text{hermitian supermanifolds}\} \to \{\text{Sets}\},$$

$$Z \mapsto (\mathcal{O}(Z) \otimes V_{\mathbb{C}})_{0,\mathbb{R}} .$$

$L(\operatorname{End}(V_{\mathbb{C}})_{\mathbb{R}})$ is the representing object for the cofunctor

$$Z \mapsto \operatorname{End}_{\mathcal{O}(Z)}(\mathcal{O}(Z) \otimes V_{\mathbb{C}})_{0,\mathbb{R}} .$$

Of course, a *hermitian supergroup* is a group object in the category of hermitian supermanifolds. The prototype of this is the *general linear hermitian supergroup* $GL(V)$; it represents the group-valued cofunctor

$$Z \mapsto GL(\mathcal{O}(Z) \otimes V_{\mathbb{C}}) := \{\text{invertible elements of } \operatorname{End}_{\mathcal{O}(Z)} (\mathcal{O}(Z) \otimes V_{\mathbb{C}})_{0,\mathbb{R}}\} .$$

If $\dim V = (m|n)$ then $\dim GL(V) = (m^2 + n^2|2mn)$.

Concerning the connections between hermitian supergroups and hermitian Lie superalgebras, there arises a delicate point: the Lie superalgebra $\mathfrak{g}$ of left invariant complex vector fields on a Lie supergroup, if equipped with the hermitian structure induced by the embedding $\mathfrak{g} \subseteq \operatorname{End}(\mathcal{O}(G))$ (cf. also 3.7. below), satisfies the law

$$\overline{[u,v]} = \epsilon(u,v)[\bar{u},\bar{v}] = -[\bar{v},\bar{u}] \tag{3.5.1}$$

and thus is not a hermitian Lie superalgebra (although it is in accordance with the second sign rule as well).

There appear two possible ways out of this difficulty: Either one rejects the definition of 2.6 and uses instead Lie superalgebras with a hermitian conjugation which satisfies (3.5.1), or one introduces the opposite hermitian structure on $\mathfrak{g}$, i. e. interchanges real and imaginary part:

$$u^+ := -\bar{u} .$$

We would like to plead for the second possibility. Both theories are obviously isomorphic; but the second one is better adapted to the habits of physical literature: the conjugation $u \mapsto u^+$ reflects the conjugation of vector fields only in a skew way, but, as shown in 2.6, it does directly reflect operator conjugation in Hilbert state space representations which often occur in physics. In A III.4 of [4], the structure constants C_{ij}^k of a finite-dimensional Lie algebra are defined by

$$[X_i, X_j] = \mathrm{i} \sum_{k=1}^{n} C_{ij}^k X_k \tag{3.5.2}$$

where $X_1, \ldots, X_n$ = are the «infinitesimal generators». The additional i in front of the sum appears also in almost any textbook on quantum field theory which gives a crash course in Lie theory, and it is caused by the wish to have the X_i represented by self-adjoint operators in unitary representations of the corresponding group (recall that if the group appears as symmetry group of a quantum theory then these self-adjoint operators describe conserved quantities). Without the i , (3.5.2) would describe an ordinary real Lie algebra $\mathfrak{g}$. If we take it as it stands, and if we call the X_i real then (3.5.2) describes instead that hermitian Lie algebra which arises from $\mathfrak{g}$ by complexifying and introducing the opposite of the obvious conjugation.

Also, the i occuring in the coordinate form of the exponential map given in [4],

$$g(a_1, \ldots, a_n) = \exp(\mathrm{i} \sum_{k=1}^{n} X_k a_k) ,$$

is caused by the same reasons. Thus, physicists have definite reasons for breaking with the traditions of mathematical literature by calling real what a mathematician would call imaginary, and vice versa.

A *vector bundle sheaf* is a locally free sheaf of hermitian $\mathcal{O}$-modules; this notion takes over the role which is played by locally free $\mathcal{O}$-modules (alias graded section modules of vector bundles, cf. [10]) in traditional supergeometry.

A *vector bundle* $E \to X$ is defined as in the traditional situation (cf. [10]) as bundle with fibre $L(\mathbb{R}^{m|n})$ and structure group $GL(\mathbb{R}^{m|n})$. Then the fibrewise linear superfunctions on E form a vector bundle sheaf $\underline{\Gamma}^*(.,E)$ on B . Its dual $\underline{\Gamma}(.,E)$ is the *associated graded section module* of E; and one recovers the sheaf $\Gamma(.,E)$ of the «geometrical» sections as the even, *real* part of $\underline{\Gamma}(.,E)$.

As to be expected, one has an equivalence between the category of vector bundles and the category of vector bundle sheaves (morphisms in this category being real, even $\mathcal{O}$-module homomorphisms).

3.6. The supersymmetry group

The Lie superalgebra of simple supersymmetry (cf. e. g. [16]) determines a unique connected, simply connected hermitian Lie supergroup which we call the *supersymmetry group*. In the following, we describe its structure in brief, without any proof. For more information, cf. [16].

We start with the hermitian vector space V given by $V_{0,\mathbf{R}} := \mathbf{R}^4$, $V_1 := \Psi \oplus \bar{\Psi}, \Psi := \mathbb{C}^2$. Let (x^μ) be the standard linear coordinate system on $\mathbf{R}^4$, and let (θ^α) $(\alpha = 1, 2)$ be the standard complex linear coordinate system on Ψ. Then $(x^\mu|\theta^\alpha)$ is a chiral coordinate system on $G := L(V)$.

REMARK. In the physical context, Ψ is the representation space of the fundamental representation of $SL(2, \mathbb{C})$. Moreover, $\Psi \otimes \bar{\Psi}$ carries an obvious hermitian structure, and V_0 is nothing but the real part of this. Physically, Ψ and $\bar{\Psi}$ are the spaces of left- and right-handed spinors, respectively; V_1 is the space of Dirac spinors, and $V_{1,\mathbf{R}}$ is the subspace of Majorana spinors. Finally, V_0 is the Minkowski space.

Let $(y^\mu|\eta^\alpha)$ be the same chiral coordinate system on a copy of G. Then, the product morphism $m : G \times G \to G$ is determined by

$$m^*(x^\mu|\theta^\alpha) = (x^\mu + y^\mu + \mathrm{i}\,\theta\sigma^\mu\bar{\eta} - \mathrm{i}\,\eta\sigma^\mu\bar{\theta}|\theta^\alpha + \eta^\alpha).$$

This formula is the usual one in physical literature, and, due to the remarks above, this is a well-defined morphism. The point is that one would run into trouble if working with the traditional calculus since then $m^*(x^\mu)$ would not be real. If one tried to form the connected, simply connected Lie supergroup belonging to the Lie superalgebra of $N = 1$ supersymmetry one would get the same formula as above but without the i. A physicist would reject the emerging formula.

3.7. Vector fields, forms

A *vector field* on X is a derivation of $\mathbb{Z}_2$-graded algebra sheaves $\delta : \mathcal{O} \to \mathcal{O}$. Analogously to the traditional situation, the vector fields form a vector bundle sheaf $\mathcal{X}$ on X; in accordance with 2.3, conjugation is given by $\bar{\delta}(u) = \epsilon(\delta, u)\overline{\delta(\bar{u})}$ for $u \in \mathcal{O}$. If $(x|\xi)$ is a coordinate system on U then $\mathcal{X}|_U$ has the base $(\partial/\partial x|\partial/\partial\xi)$. Note that while the $\partial/\partial x_i$ are real, the $\partial/\partial\xi_j$ are imaginary (this does not violate the sign rules since $\partial/\partial \ldots$ is a usual but somewhat abusive notation, not a multilinear expression).

The *exterior differential* $d : \mathcal{O} \to \Omega^1 := \Pi\,\mathrm{Hom}_{\mathcal{O}}(\mathcal{X}, \mathcal{O})$ is defined by $\langle\delta, \Pi\,du\rangle = \mathrm{i}\,\delta(u)$ for $u \in \mathcal{O}$, $\delta \in \mathcal{X}$ (cf. [10] for the traditional treatment). We introduced the additional i in order to ensure that $d = \bar{d}$ is real (otherwise, the usual relation $d = \partial + \bar{\partial}$ on complex manifolds would have to be modified). Since d is odd it follows that for real $u \in \mathcal{O}$, du is real and imaginary if u is even and odd, respectively. The further development of the theory of (pseudo-) differential forms exactly parallels the traditional treatment.

4. RELATION TO COMPLEX SUPERMANIFOLDS

4.1. Complex structures

In an obvious sense, the material of 2.4 carries over from modules to module sheaves. A vector bundle sheaf which is equipped with a complex structure on it is called a *complex vector bundle sheaf*; the apparently strange terminology is justified by the fact that in hermitian geometry, these structures play the same role as complex vector bundles in traditional supergeometry as well as in non-super geometry. In accordance to the remarks in 2.4, for knowing a complex vector bundle sheaf E of rank $(2m|2n)$ it is sufficient to know only the arising locally free $\mathcal{O}$-module sheaf E_+ of rank $(m|n)$, and this may be prescribed arbitrarily. In fact, E_+ is that sheaf with which one would describe a complex vector bundle in the traditional or classical situation.

If X is a hermitian supermanifold an *almost complex structure on* X is a complex structure on Ω^1. We call this a *complex structure* if involutivity holds in its usual form: $d\Omega^1_- \subseteq \Omega^1 d\Omega^1_-$. We call a hermitian supermanifold which is equipped with a complex structure a *complex supermanifold.* We also introduce the usual notation (cf. 2.4)

$$\Omega^{i,j} := S^{i,j}\Omega^1 .$$

REMARK. (1) In contrast to the traditional setting, the $\Omega^{i,j}$ are not vector bundle sheaves in itself. A similar remark also applies to RC and CR structures (cf. [11], [12]) on supermanifolds. As far as I know, this is the only significant instance where something goes in traditional supergeometry but not in the hermitian setting.

(2) The theory of complex manifolds *per se* (i.e. of ringed spaces locally isomorphic to $(U, \mathcal{A}[\zeta_1, \ldots, \zeta_n]), U \subseteq \mathbb{C}^m$ open, $\mathcal{A}$ is the sheaf of holomorphic functions on U) remains unchanged. Indeed, hermitian conjugation makes analytic things antianalytic, and thus has no place within the analytic realm. What changes (but only in details) is the relation of complex manifold to their underlying real supermanifolds: every complex manifold in the sense above determines an underlying traditional real supermanifold (cf. [11]) as well as an underlying hermitian supermanifold, and it is the relation to the latter one we are going to describe in the following.

The following material exactly parallels the traditional treatment (cf. [11]): given an almost complex supermanifold X, one introduces operators $\partial, \bar{\partial}$ by

$$\partial : \Omega^{i,j} \xrightarrow{d} \Omega^{i+j+1} \xrightarrow{\text{projection}} \Omega^{i+1,j},$$

$$\bar{\partial} : \Omega^{i,j} \xrightarrow{d} \Omega^{i+j+1} \xrightarrow{\text{projection}} \Omega^{i,j+1} .$$

By linearity, they extend onto $\Omega := \bigoplus_{k\geq 0} \Omega^k$, and we have

(4.1.2) $$\bar{\partial}\bar{\omega} = (-1)^{|\omega|}\overline{\partial\omega} \quad \text{for} \quad \omega \in \Omega .$$

Indeed, using that d is real we find for $\omega \in \Omega^{ij}$

$$\begin{aligned}\overline{\bar{\partial}\bar{\omega}} &= \overline{(d\bar{\omega})_{i,j+1}} = (\overline{d\bar{\omega}})_{j+1,i} \\ &= (-1)^{|\omega|}(d\omega)_{j+1,i} = (-1)^{|\omega|}\partial\omega .\end{aligned}$$

Thus, $\bar{\partial}$ is indeed the hermitian conjugated operator to ∂, as the notation suggests. By type comparison, ∂ and $\bar{\partial}$ are odd derivations of the sheaf of $\mathbb{Z}_2$-commutative algebras Ω.

Warning: if a non-super complex manifold is to be considered as supermanifold, one has to be careful: in form degrees $4k+2$, $4k+3$, hermitian conjugation differs from usual complex conjugation by a minus sign. This is the source for the sign in (4.1.2), which in the usual non-super treatment with ordinary complex conjugation is absent.

Now a given almost complex structure is involutive iff $\partial + \bar{\partial} = d$. If this is satisfied then ∂, $\bar{\partial}$, and d commute (in the Lie superalgebra of endomorphisms of Ω).

The superversion of the Newlander-Nirenberg holds (cf. [11] for the proof in the traditional situation; it carries over to the hermitian one). That is, given a complex supermanifold, there exists locally always a sequence $(z|\zeta)$ of sections of $\mathcal{O}$ such that

(1) $(x, y|\xi, \eta) := (\operatorname{Re}(z), \operatorname{Im}(z)|\operatorname{Re}(\zeta), \operatorname{Im}(\zeta))$ is a local real coordinate system, and (2) $\bar{\partial}z_i = \bar{\partial}\zeta_j = 0$ for all i, j.

We then call $(z|\zeta)$ an *analytic coordinate system*. As in the traditional setting, we have

$$\begin{aligned}\bar{\partial} &= \sum d\bar{z}_i\partial/\partial\bar{z}_i + \sum d\bar{\zeta}_j\partial/\partial\bar{\zeta}_j, \\ \partial/\partial\bar{z}_i &:= (\partial/\partial x_i + \sqrt{-1}\partial/\partial y_i)/2, \\ \partial/\partial\bar{\zeta}_j &:= (\partial/\partial\xi_j + \sqrt{-1}\partial/\partial\eta_j)/2,\end{aligned}$$

and the sheaf of *holomorphic superfunctions* $\mathcal{A} := \operatorname{Ker}\{\bar{\partial} : \mathcal{O} \to \Omega^{0,1}\}$ is a subalgebra sheaf of $\mathcal{O}$. In local coordinates,

$$\mathcal{A} = \{f(z,\zeta) = \sum f_\mu(z)\zeta^\mu; \quad f_\mu(z) \text{ holomorphic}\}.$$

Note, however, that $\mathcal{A}$ is not a sheaf of hermitian vector spaces. In fact, its intersection with its conjugate $\bar{\mathcal{A}} = \operatorname{Ker}\{\partial : \mathcal{O} \to \Omega^{1,0}\}$ (sheaf of *antiholomorphic superfunctions*) consists of constants only. Note also that while in the traditional situation the map $\mathcal{A} \to \bar{\mathcal{A}}, f \mapsto \bar{f}$, was an isomorphism, it is now an antiisomorphism: $\overline{fg} = \bar{g}\bar{f} = \epsilon(f,g)\bar{f}\bar{g}$.

4.2. $\bar\partial$-connections

Let X be a fixed complex supermanifold, E a complex vector bundle sheaf. A $\bar\partial$-connection $\bar\nabla$ on E is an odd map of sheaves of vector spaces

$$\bar\nabla : E_+ \to E_+ \otimes \Omega^{0,1}$$

(note the difference as compared to the traditional case in the domain of definition) which satisfies the usual Leibniz rule: $\bar\nabla(eu) = \bar\nabla(e)u + (-1)^{|e|}e\bar\partial u$ for $e \in E, u \in \mathcal{O}$. Quite analogous to the classical case, $\bar\nabla$ determines higher covariant derivatives $\bar\nabla^j : E_+ \otimes \Omega^{0,j} \to E_+ \otimes \Omega^{0,j+1}$ and a curvature $R := \bar\nabla^1\bar\nabla : E_+ \to E_+ \otimes \Omega^{0,2}$ which is an $\mathcal{O}$-linear map. For $R = 0, \bar\nabla$ is called *flat.*

Now let F_{an} be a locally free $\mathcal{A}$-module of rank $m|n$. We associate with F_{an} a hermitian $\mathcal{O}$-module $F := F_{\text{an}} \otimes \mathcal{O} \oplus \overline{F_{\text{an}} \otimes \mathcal{O}}$ (here $\overline{F_{\text{an}} \otimes \mathcal{O}}$ is the «exterior» conjugate of $F_{\text{an}} \otimes \mathcal{O}$, cf. 2.4); conjugation acts as the notation suggests. Then F is locally free of rank $2m|2n$, and it carries a canonical complex structure given by $F_+ = F_{\text{an}} \otimes \mathcal{O}$. Moreover, F carries a unique $\bar\partial$-connection $\bar\partial$ (usual notation) which annihilates the subsheaf F_{an}. In fact, $\bar\partial$ is flat, and its kernel is exactly F_{an}. The passage from F_{an} to F corresponds to the traditional fact that any holomorphic vector bundle can be viewed as a real vector bundle of doubled rank. The «doubling» of degrees of freedom looks somewhat bewildering. However, we shall see below, when treating tangent bundles, that it makes its appearance in an implicite way already on the classical level.

Let E_{an}, F_{an} be locally free $\mathcal{A}$-modules. Then we have canonical isomorphisms (cf. 2.4 for the notations)

$$E_{\text{an}} \otimes F_{\text{an}} = (E \otimes_{\mathcal{O},J} F)_{\text{an}}, \quad S^k_{\mathcal{A}} E_{\text{an}} = (S^k_{\mathcal{A},J} E)_{\text{an}},$$
$$\text{Hom}_{\mathcal{A}}(E_{\text{an}}, F_{\text{an}}) = \text{Hom}_{\mathcal{A},J}(E, F)_{\text{an}}, \quad E_{\text{an}}{}^* = (E^*)_{\text{an}}.$$

This justifies the settings of 2.4.

As in the traditional situation (cf. [11]), the assignment $E_{\text{an}} \mapsto (E, J, \bar\partial)$ can be inverted:

THEOREM. *Let be given a complex vector bundle E and a flat $\bar\partial$-connection $\bar\nabla$ on it. Then $E_{\text{an}} := \text{Ker}(\bar\nabla)$ (sheaf analytic sections of E) is a locally free $\mathcal{A}$-module, and the arising map $E_{\text{an}} \otimes \mathcal{O} \to E_+$ is isomorphic.* Q.E.D. ∎

For example, $\bar\partial : \Omega^{i,0} \to \Omega^{i,1} = \Omega^{i,0} \otimes \Omega^{0,1}$ is a flat $\bar\partial$-connection on $S^i_J\Omega^1 = \Omega^{i,0} \oplus \Omega^{0,i}$; its kernel Ω^i_{an} is the sheaf of *analytic i-forms*.

By duality, we get a complex structure as well as a flat $\bar\partial$-connection on $\mathcal{X}$. If $(z|\zeta)$ is an analytic coordinate system then $(\partial/\partial z|\partial/\partial\zeta)$ is a base of the $\mathcal{A}$-module sheaf of *analytic vector fields* $\mathcal{X}_{\text{an}}$.

4.3. Hermitian metrics

Let E be a complex vector bundle sheaf on a hermitian supermanifold X. A *hermitian metric* on E is an $\mathcal{O}$-bilinear even map

$$g : E_- \otimes E_+ \to \mathcal{O} \tag{4.3.1}$$

which moreover satisfies for all $e, f \in E_+$

(1) $\overline{g(\bar{e}, f)} = g(\bar{f}, e)$, i. e. g is real;

(2) The arising $\mathcal{O}$-homomorphism $g' : E_- \to E_+^*, \bar{e} \mapsto g(\bar{e}, .)$, of E_- into the dual of E_+ is an isomorphism;

(3) $\beta(g(\bar{e}, e)) \geq 0$ for $e \in E_+$, i.e. g is *positive definite*

(we recall that β is the pullback from superfunctions to functions on the underlying manifold). Note that by (1), $\beta(g(\bar{e}, e))$ is real-valued, and thus (3) makes sense. The reader will note that our treatment closely follows that of 2.5.

Let $\hat{E}_+ := E_+ \otimes_{\mathcal{O}} C^\infty_{\mathbb{C}}$, and let $E_+ \to \hat{E}_+, e \mapsto \hat{e}$ be the canonical projection. It is known that this is surjective on the level of global sections (cf. [10], 7.9). Now any map (4.3.1) gives rise to a skew-linear form on $\hat{E}_+$:

$$g : E_- \otimes E_+ \to \mathcal{O},\ (\hat{e}, \hat{f}) \mapsto \beta(g(\bar{e}, f)) , \tag{4.3.2}$$

and, by [10], Cor. 7.8, the requirement (2) is equivalent with the arising $C^\infty_{\mathbb{C}}$-homo–morphism $\hat{g}' : \hat{E}_- \to \hat{E}_+^*, \bar{\hat{e}} \mapsto \beta(g(\bar{e}, .))$, being an isomorphism. This is in turn equivalent with saying that (4.3.2) is non-degenerate.

In particular, if g is a hermitian metric then (4.3.2) is a hermitian metric on $\hat{E}_+$ in the usual sense. It also follows easily, using a partition of unity argument, that any complex vector bundle can be equipped with a hermitian metric. Also, the restriction of a hermitian metric onto a complex subbundle (with an obvious definition of this notion) is a hermitian metric on the latter, too.

Given a hermitian metric g on the complex vector bundle E we may construct two bilinear maps

$$g_\pm : E \otimes E \to \mathcal{O},$$
$$g_+(a + \bar{b}, c + \bar{d}) := \frac{1}{2}(g(\bar{b}, c) + \epsilon(d, a) g(\bar{d}, a)),$$
$$g_-(a + \bar{b}, c + \bar{d}) := \frac{i}{2}(g(\bar{b}, c) - \epsilon(d, a) g(\bar{d}, a))$$

for $a, b, c, d \in E_+$. The following assertions follow by direct computation: If we equip $E \otimes E$ with the tensor product hermitian structure then both g_+ and g_- are real; while g_+ is $\mathbb{Z}_2$symmetric, g_- is $\mathbb{Z}_2$-alternating, and the identity

$$-g_\pm(Je, Jf) = g_\pm(e, f) = g_\mp(e, Jf) = -g_\mp(Je, f)$$

holds for all $e, f \in E$. As it will become clear below, one should view g_+ and g_- as hermitian analoga of the usual «real» and «imaginary part» of the metric.

Indeed, let g be a hermitian metric on X. As in the classical case, g determines a *fundamental form* $\omega \in \Omega^1$ which is characterized (cf. [10], 8.9 and Prop. 7.43.2) by

$$i_\xi i_\eta(\omega) = (-1)^{|\xi|} g_-(\xi, \eta)$$

for $\xi, \eta \in \mathcal{X}$. (i_ξ is the interior derivative. The sign factor makes the right-hand side $\mathbb{Z}_2$-symmetric in $\Pi\xi$, $\Pi\eta$; otherwise, the definition would not be correct.) Since $g_-(\xi, \eta) = 0$ whenever $\xi, \eta \in \mathcal{X}_+$ or $\xi, \eta \in \mathcal{X}_-$, we have

$$\omega \in \Omega^{1,1},$$

as to be expected.

If $e^1, \dots, e^{m+n}$ is a basis of $\Omega^{1,0}$ which is left dual to a unitary basis of $\mathcal{X}$ (cf. 4.4) then one easily computes

$$\omega = \frac{\sqrt{-1}}{2} \sum (-1)^{|e^i|} e^i \overline{e^i}.$$

For $n = 0$ we recover the classical formula. Note that ω is imaginary (recall that if we view a manifold as hermitian supermanifold then hermitian conjugation on two forms is the opposite of traditional conjugation; thus, there is no contradiction to classical theory).

4.4. Unitary bases

Assume that the $\mathcal{O}$-module E_+ has the basis $(e) = (e_1, \dots, e_{k+l})$, and choose elements $g_{ij} \in \mathcal{O}$. Then the setting

$$g(\overline{e_i}, e_j) := g_{ij}$$

defines a hermitian metric on E iff $\overline{g_{ij}} = g_{ji}$ for all i, j (no sign). In particular, the setting

$$(4.4.1) \qquad g(\overline{e_i}, e_j) := \delta_{ij}$$

defines a hermitian metric on E. Explicitely,

$$g(\overline{\sum e_j v_j}, \sum e_k w_k) = \sum \overline{v_j} w_j.$$

Conversely:

PROPOSITION. *Given a complex vector bundle with hermitian metric, there exists locally always a unitary basis, i.e. a basis* (e) *of* E_+ *which satisfies (4.4.1). In fact, any unitary basis* $\hat{e}_1, \dots, \hat{e}_{m+n}$ *of* $\hat{E}$ *can be lifted to a unitary basis* $e_1, \dots, e_{m+n}$.

Proof. By our remarks above, we can lift $\hat{e}_1, \dots, \hat{e}_{m+n}$ to a sequence $e'_1, \dots, e'_{m+n} \in E_+$. We apply Gram-Schmidt orthogonalization onto this sequence: If $e_1, \dots, e_k$ are already constructed set $e'' := e'_{k+1} - \sum_{i=1}^{k} e_i g(\bar{e}_i, \bar{e}'_{k+1})$, $f := g(\overline{e''}, e'') \in \mathcal{O}$. Then $\beta(f) = 1$, and hence, by the «functional calculus» (cf. [10], 3.5), $f^{-1/2} \in \mathcal{O}$ is well-defined. Set $e_{k+1} := f^{-1/2} e''$. Then the elements $e_1, \dots, e_{k+1}$ satisfy (4.4.1). By induction, one easily has $(\hat{e_i}) = \hat{e}_i$ for all i, justifying the notation. By [10], Cor. 7.8, the sequence $e_1, \dots, e_{m+n}$ is indeed a basis of E_+. Q.E.D. ∎

Let E, F be complex vector bundles with hermitian metrics g_E, g_F. The composite

$$(E_- \otimes F_-) \otimes (E_+ \otimes F_+) \xrightarrow{\text{interch.}} E_- \otimes E_+ \otimes F_- \otimes F_+ \xrightarrow{g_E \otimes g_F} \mathcal{O}$$

is a hermitian metric on $E \otimes_J F$ (cf. 2.4). For $e, e' \in E_+$, $f, f' \in F_+$ we have $g(\overline{e \otimes f}, e' \otimes f') = (-1)^{(|e|+|e'|)|f|} g_E(\bar{e}, e') g_F(\bar{f}, f')$. If (e_i), (f_j) are unitary bases of E_+, F_+, respectively, then $(e_i \otimes f_j)$ is a unitary base of $(E \otimes_J F)_+$.

Also, ΠE carries a hermitian metric $g(\overline{\Pi e}, \Pi f) := g(\bar{e}, f)$; (Πe_i) is a unitary basis of it.

One can also introduce a hermitian metric on $S^k_J E$: the isomorphism g' of 4.3 gives rise to an isomorphism $S^k g' : S^k E_{-} \to S^k(E_+^*) = (S^k E_+)^*$ (cf. [10], Prop. 7.46), and we may set

$$g(\bar{\mu}, \nu) := S^k g'(\bar{\mu})(\nu).$$

Thus, a unitary base of $S^k_J E$ is given by all monomials

$$e^{(\mu,\nu)}/\mu! := e_1^{\mu_1} \dots e_m^{\mu_m} e_{m+1}^{\nu_1} \dots e_{m+n}^{\nu_n}/\mu!$$

with $\mu \in \mathbf{Z}_+^m$, $\nu \in \mathbf{Z}_2^n$, $|\mu| + |\nu| = k$ (this strongly reminds of the usual basis in Fock space, and this is of course no accident).

Let us try to equip the dual bundle E^* with a hermitian metric. g induces an isomorphism

$$g'' : E_+ \to E_-^*, \quad g''(e)(\bar{f}) = \epsilon(e, f) g(\bar{f}, e).$$

The direct sum of g'' and g' is an isomorphism

$$h : E \to E^* = E_+^* \oplus E_-^*,$$

which, however, anticommutes with the complex structures. We try to carry over the hermitian metric onto E^* :

$$g(h(e),h(\bar{f})) := \epsilon(e,f)g(\bar{f},e).$$

Thus, $g(h(e_i),h(\overline{e_j})) := (-1)^{|e_i|}\delta_{ij}$. Unfortunately, the sign spoils the positive definiteness, and, as far as I know, no redefinition can remove it. It seems impossible to equip E^* with an invariantly defined hermitian metric (this is embarassing since it makes the notion «Kähler supermanifold» ambiguous: should we take a hermitian metric on Ω^1 or on $\mathcal{X}$?).

4.5. The associated connection

A *connection* on a vector bundle sheaf E is a real, odd map $\nabla : E \to E \otimes \Omega^1$ which satisfies the usual Leibniz rule:

$$\nabla(eu) = \nabla(e)u + (-1)^{|e|}edu \tag{4.5.1}$$

for $e \in E, u \in \mathcal{O}$. ∇ determines higher covariant derivatives $\nabla^i : E \otimes \Omega^i \to E \otimes \Omega^{i+1}$ for all $i \geq 0$ and a curvature $R := \nabla^1\nabla \in \mathrm{End}(E) \otimes \Omega^2$ (cf. [11] for the classical treatment). One easily checks that the ∇^i are real while R is imaginary (this apparently contradicts the classicl theory; but recall that hermitian conjugation on Ω^2 is the opposite of traditional conjugation). Given a real base $e_1,\ldots,e_{m+n}$ of E and forms $\omega_{ij} \in \Omega^1$, the setting

$$\nabla e_i = \omega_{ij}e_j$$

determines a connection on E iff

$$\overline{\omega_{ij}} = (-1)^{|e_i|+|e_i||e_j|}\omega_{ij} \tag{4.5.1}$$

for all i,j (here and in the following, we use the summation convention; note that indices appearing in the exponents of sign factors generate no extra summation). In particular, if all e_i are even then all ω_{ij} have to be real, as in the classical theory.

As to be expected, one has:

PROPOSITION. *Let E be a complex vector bundle sheaf which is equipped with a flat $\bar{\partial}$-connection $\bar{\partial}$ and a hermitian metric g. There exists a unique connection ∇ on E such that for $e,f \in E_+$ we have*

$$\begin{aligned} &(\nabla e)_{0,1} = \bar{\partial}e, \\ &dg(\bar{e},f) = (-1)^{|e|}(g(\overline{\nabla e},f) + g(\bar{e},\nabla f)) \end{aligned} \tag{4.5.2}$$

(the definition of the terms on the r. h. s. is obvious).

Proof. Suppose that ∇ exists, and set $\nabla_{1,0}e := \nabla e - \bar\partial e$ for $e \in E_+$. Taking the $0,1$-part in (4.5.2) we find

$$\bar\partial g(\bar e, f) = (-1)^{|e|}(g(\overline{\nabla_{1,0}e}, f) + g(\bar e, \bar\partial f)),$$

i.e.

$$(4.5.3) \qquad \begin{aligned} &g(\overline{\nabla_{1,0}e}, f) = F(\bar e, f),\\ &F(\bar e, f) := (-1)^{|e|}\bar\partial g(\bar e, f) - g(\bar e, \bar\partial f). \end{aligned}$$

This being true for all $f \in E_+$, $\nabla_{1,0}e$ is uniquely determined, and, using also the reality of ∇, the uniqueness assertion follows. Turning to existence we first note that for fixed $e \in E_+$, $F(\bar e, .)$ is $\mathcal{O}$-linear: for $u \in \mathcal{O}$, we have

$$\begin{aligned} F(\bar e, fu) &= (-1)^{|e|}\bar\partial(g(\bar e, f)u) - g(\bar e, \bar\partial f)u - (-1)^{|f|}g(e, f)\bar\partial u\\ &= (-1)^{|e|}\bar\partial g(\bar e, f)u - g(\bar e, \bar\partial f)u = F(\bar e, f)u. \end{aligned}$$

Therefore it determines a homomorphism $F(\bar e, .) : E_+ \to \Omega^{0,1}$, and hence there is a unique element $\nabla_{1,0}e$ satisfying (4.5.3) for all f.

Thus we have a well-defined map $\nabla_{1,0} : E_+ \to E_+ \otimes \Omega^{1,0}$. Moreover,

$$\begin{aligned} g(\overline{\nabla_{1,0}(eu)}, f) &= F(\overline{eu}, f) = (-1)^{|u|+|e|}\bar\partial(\bar u g(\bar e, f)) - \bar u g(\bar e, \bar\partial f)\\ &= (-1)^{|e|}\overline{\partial u}g(\bar e, f) + (-1)^{|e|}\bar u\bar\partial g(\bar e, f) - \bar u g(\bar e, \bar\partial f)\\ &= (-1)^{|u|+|u||e|}g(\bar e \otimes \overline{\partial u}, f) + \bar u F(\bar e, f)\\ &= g((-1)^{|e|}\overline{e \otimes \partial u} + \overline{(\nabla_{1,0}e)u}, f), \end{aligned}$$

and hence

$$(4.5.4) \qquad \nabla_{1,0}(eu) = (-1)^{|e|}e \otimes \partial u + (\nabla_{1,0}e)u.$$

Therefore $\nabla := \nabla_{1,0} + \bar\partial : E_+ \to E_+ \otimes \Omega^1$ satisfies the Leibniz rule (4.5.1). Extending it onto E by setting $\nabla\bar e := (-1)^{|e|}\overline{\nabla e}$ we get the connection ∇ wanted. Q.E.D. ∎

Let $e_1, \ldots, e_{k+l}$ be a holomorphic base of E_+. Let $g(\overline{e_i}, e_j) = g_{ij}$, and let (g^{kl}) be the inverse matrix: $\sum g^{ik}g_{kj} = \delta^i_j$. Thus, $g(\overline{e_i}, e_k g^{kj}) = \delta^j_i$. Hence, using (4.5.4),

$$\begin{aligned} &F(\overline{e_k g^{kj}}, e_i) = 0, \quad \nabla_{1,0}(e_k g^{kj}) = 0,\\ &\nabla e_j = \nabla_{1,0}e_j = \nabla_{1,0}(e_k h^{ki}h_{ij}) = (-1)^{|e_k|+|h_{ki}|}e_k h^{ki}\partial h_{ij}. \end{aligned}$$

Thus, the final result is:

$$\nabla e_j = (-1)^{|e_i|} e_k g^{ki} \partial g_{ij} .$$

The curvature is given by

$$Re_j = \nabla\nabla e_j = (-1)^{|e_i|+|e_k|} e_k \bar{\partial} g^{ki} \partial g_{ij} .$$

Hence

$$R \in E \otimes \Omega^{1,1} .$$

If we start with a unitary base $e_1, \ldots, e_{k+l}$ instead of a holomorphic one, and if $\bar{\partial} e^i = \omega_{ij} e_j$ with $\omega_{ij} \in \Omega^{0,1}$, we find

$$\nabla e_i = (\omega_{ij} + (-1)^{|e_i|(|e_j|+1)} \overline{\omega_{ij}}) e_j .$$

4.6. Concluding remarks

(1) As we just saw, that classical material which is algebraic in its very nature carries over to the super-situation. On the other hand, it is not so clear how to define*Kähler supermanifolds* properly (should we take a hermitian metric on the tangent or on the cotangent bundle?), and what these are good for. In particular, does there exist a super Hodge theory, and what are its main Theorems? The essential ingredient of Hodge theory on classical Kähler manifolds is *positivity*, and this notion makes genuine difficulties in the super context. Certainly, the passing to the hermitian setting is a step in the right direction, as the consistency of the notion «super Hilbert space» shows. But, may be, it is only a step towards the wall which separates supermanifolds from the realm of classical Hodge theory.

The source of the trouble is that one misses a scalar product on superfunctions which is both invariant and positive definite: one can associate with g an invariant volume form μ , and the resulting scalar product $(u, v) \mapsto \int \bar{u} v \mu$ on functions has the right symmetry properties but it is not positive definite.

Also, it is not yet clear what the supervariants of the Theorems of classical Hodge theory are. For instance, while on a compact Kähler manifold every holomorphic form is closed, this is certainly false in the super case (look at $L(\mathbb{C}^{0|n}))$.

If an interesting super Hodge theory exists it is clear from the observations above that it cannot be a simple variant of classical Hodge theory.

(2) On the other hand, I conject that if V is a Kähler potential (i.e. $\partial\bar{\partial} V = \omega$) then

(4.6.1) $$(\bar{u}, v) \mapsto \int \bar{u} v \exp(\mathrm{i} V) \mu$$

is positive definite as long as u, v are analytic superfunctions (this would generalize the fermionic variant of the Bargman-Fock construction; cf. [1]). For globalizing this, one should take u, v to be analytic sections of a line bundle with hermitian metric the curvature of which is exactly the fundamental form ω. Then we are in the realm of geometric quantization. The difficulty, however, is to give (4.6.1) a precise meaning: a priori, integrals over volume forms are defined only for compact support, which is incompatible with u, v being analytic.

(3) Resuming, let us list some items of the dictionary between the notions of traditional supergeometry and the hermitian language:

Traditional Notion	*Hermitian Pendant*
real $\mathbb{Z}_2$-graded v.s. $\mathbb{R}^{m\|n}$	hermitian v.s. $\mathbb{C}^{m\|n}$ with componentwise conjugation
complex $\mathbb{Z}_2$-graded v.s. $\mathbb{C}^{m\|n}$	hermitian v.s. with complex structure $\mathbb{C}^{m\|n} \otimes_{\mathbb{R}} \mathbb{C}$, with complex vector space structure defined by first factor, J acts as i on second factor
$\mathbb{Z}_2$-graded $\mathbb{R}$-algebra	$\mathbb{Z}_2$-graded hermitian algebra
module over it	hermitian module over it
$\mathbb{Z}_2$-graded $\mathbb{C}$-algebra	- remains -
real Lie superalgebra	hermitian Lie superalgebra
complex Lie superalgebra	- remains -
(no pendant)	super Hilbert space
superdomain/smf	hermitian superdomain/smf
linear superspace $L(V)$	$L(V)$ (cf. 3.5)
locally free $\mathcal{O}$-module	hermitian locally free module (= vector bundle sheaf)
$\otimes$, Hom, $S^k, (.)^*, \Pi$ for these	same with induced hermitian structure
connections	- same treatment -
locally free $\mathcal{O}_{\mathbb{C}}$-module	hermitian l. f. module with complex structure (= complex vector bundle sheaf)
complex $\otimes$ Hom $S^k, (.)^* \Pi$ for these	$\otimes_J$, Hom$_J$, $S^k_J, (.)^*, \Pi$
complex conjugated bundle $\bar{E}$	E with opposite complex structure
complex smf's per se (cf. 4.1)	- remain -
complex structures on smf's	- essentially same treatment -
$\bar\partial$-connections $\bar\partial : E \to E \otimes \Omega^{0,1}$	$\bar\partial$-connections $\bar\partial : E_+ \to E_+ \otimes \Omega^{0,1}$
extension of scalars $F := F_{an} \otimes \mathcal{O}_{\mathbb{C}}$	$F := F_{an} \otimes \mathcal{O} \oplus \overline{F_{an} \otimes \mathcal{O}}$ (cf. 4.2)
(could be defined, too)	hermitian metrics
real and imaginary part of these	bilinear forms g_+, g_-
(could be defined, too)	associated connection
(no pendant)	unitary bases

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